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Recursive Relations for the S-matrix of Liouville Theory

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The relation between the vertex operators of the in and out fields in Liouville theory is analyzed. This is used to derive equations for the S-matrix, from which a recursive relation for the normal symbol of the S-matrix for discrete center-of-mass momenta is obtained. Its solution is expressed as multiple contour-integrals of a generalized Dotsenko–Fateev type. This agrees with the functional integral representation of the scattering matrix of Liouville theory, which had been proposed previously.

1. Introduction and Summary

Liouville theory has been for many years the prime example of an interacting two-dimensional field theory where exact results can be obtained. It is an integrable conformal field theory with the remarkable property that conformal invariance and integrability can be maintained in the quantum theory. For thorough reviews, we refer to ref. [1, 2] and for more recent developments to refs. [3–5].

In this note, we consider quantum Liouville theory on a Minkowski cylinder,[1] more specifically its scattering matrix. The solutions of the theory are such that at asymptotic times they approach massless free fields and the scattering matrix is the overlap between asymptotic *in*- and *out*-states. There are various ways of computing exact S-matrix elements. They use the integrability of the system, which implies the existence of an infinite number of conserved charges, which can be combined into the Virasoro-modes of the energy-momentum tensor. At asymptotic times it takes the form of the (improved) energy–momentum tensor of a free massless scalar field.

We have studied the S-matrix previously in refs. [6, 7]. In particular, we have proposed a functional integral representation for the generating functional of S-matrix elements, which involves a non-local, but nevertheless simple one-dimensional classical action of a scalar field on a circle. Note that a similar functional integral representation of the S-matrix can be realized for the $SL(2,\mathbb{R})/U(1)$ coset WZW model and $SL(3,\mathbb{R})$ Toda theory.[8] In Liouville theory, we have checked this functional integral representation in various ways. First in perturbation theory at one- and two-loop order and then also to all orders in $\hbar$, but the latter only for discrete (imaginary) values of the center-of-mass momentum. In this case, the functional integral collapses to a finite-dimensional integral, the dimension depending on the discrete momentum value. In this note, we will rederive these finite-dimensional integrals in an independent way, which uses only the known relation between the asymptotic *in*- and *out*-fields. The explicit evaluation of these integrals beyond what was presented in ref. [7] remains, however, an open problem.

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[1] In contrast to the strip, there are no bound states on the cylinder.

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2. Asymptotic Fields of Liouville Theory

In the first part of this section, we introduce the asymptotic fields of the classical Liouville theory and analyze the relation between the *in*- and *out*-fields, briefly summarizing some results of refs. [6, 7]. In the second part of the section, we introduce exponentials of the asymptotic field operators as primary fields and present the quantum analog of the classical relation between the asymptotic fields.

2.1. Classical Asymptotic Fields

We study the scattering problem in Liouville theory on a cylindrical spacetime with coordinates $\sigma \in \mathbb{S}^1$ and $\tau \in \mathbb{R}^1$. The dynamics of the system are governed by Liouville's equation

$$\partial_{\bar{x}}\partial_x \Phi + \mu^2\, e^{2\Phi} = 0, \tag{1}$$

where $x = \tau + \sigma$ and $\bar{x} = \tau - \sigma$ are the chiral coordinates and μ^2 is a positive constant.

At asymptotic times, $\tau \to \mp\infty$, the Liouville field becomes a free massless field. The asymptotic fields have the standard Fourier mode expansions

$$
\Phi_{\text{in}} = q + p\,\tau + i\sum_{n\neq 0}\left(\frac{a_n}{n}\,e^{-inx} + \frac{\bar a_n}{n}\,e^{-in\bar x}\right),
$$
$$
\Phi_{\text{out}} = \tilde q + \tilde p\,\tau + i\sum_{n\neq 0}\left(\frac{b_n}{n}\,e^{-inx} + \frac{\bar b_n}{n}\,e^{-in\bar x}\right), \tag{2}
$$

which can be written as a sum over chiral and anti-chiral parts. It is important to note that the *out*-field always differs from the *in*-field. In particular, the momentum zero mode of the *in*-field is positive, $p>0$, while $\tilde p = -p < 0$, similarly to the particle dynamics in the exponential potential, which corresponds to the homogeneous Liouville field $\partial_\sigma \Phi(\tau,\sigma) = 0$.

By the general solution of the Liouville equation, one can find the explicit form of the exponential of the *out*-field $e^{-\Phi_{\text{out}}}$ in terms of the *in*-field

$$
e^{-\Phi_{\text{out}}(x,\bar x)} = \mu^2\, e^{-\Phi_{\text{in}}(x,\bar x)} \int_{-\infty}^{x} dy \int_{-\infty}^{\bar x} d\bar y\, e^{2\Phi_{\text{in}}(y,\bar y)}. \tag{3}
$$

Another remarkable relation is the form of the Liouville field exponential $e^{-\Phi}$ in terms of the exponentials of the asymptotic fields

$$
e^{-\Phi(x,\bar x)} = e^{-\Phi_{\text{in}}(x,\bar x)} + e^{-\Phi_{\text{out}}(x,\bar x)}. \tag{4}
$$

This relation, together with (3), defines a parameterization of the interacting field Φ in terms of the asymptotic free field Φ_{in}.

If we switch off the oscillator modes and consider the homogeneous *in*-field consisting of only the zero-modes $\Phi_{\text{in}} = q + p\tau$, then by (3), the *out*-field is also homogeneous, $\Phi_{\text{out}} = \tilde q + \tilde p \tau$, with $\tilde q = -q + \log(p^2/\mu^2)$ and $\tilde p = -p$. Therefore, if all a-modes vanish, then the b-modes are also in the vacuum sector.

One can similarly parameterize Φ by the *out*-field, using the expression of the *in*-field in terms of the *out*-field, obtained by the inversion of (3)

$$
e^{-\Phi_{\text{in}}(x,\bar x)} = \mu^2\, e^{-\Phi_{\text{out}}(x,\bar x)} \int_{x}^{\infty} dy \int_{\bar x}^{\infty} d\bar y\, e^{2\Phi_{\text{out}}(y,\bar y)}. \tag{5}
$$

Note that the conditions $p>0$ and $\tilde p < 0$ provide convergence of the integrals in (3) and (5), respectively. Below, we use (3).

In the Hamiltonian formulation, Liouville theory has the standard canonical structure and the parametrization of the Liouville field in terms of the *in* (or the *out*) field induces the canonical structure for the asymptotic fields. Specifically, the analysis of the canonical symplectic form of Liouville theory shows that the Fourier modes of the asymptotic fields satisfy the canonical Poisson brackets relations[6]

$$
\{p,q\} = 1, \quad \{a_m, a_n\} = \frac{i}{2}\, m\delta_{m,-n}, \quad \{\bar a_m, \bar a_n\} = \frac{i}{2}\, m\delta_{m,-n}, \tag{6}
$$

$$
\{\tilde p,\tilde q\} = 1, \quad \{b_m, b_n\} = \frac{i}{2}\, m\delta_{m,-n}, \quad \{\bar b_m, \bar b_n\} = \frac{i}{2}\, m\delta_{m,-n}. \tag{7}
$$

One can also check by direct computation that (7) follows form (3) and (6).

Indeed, the canonical Poisson brackets (6) are equivalent to the causal relation between the *in*-field exponentials

$$
\{e^{-\Phi_{\text{in}}(x,\bar x)}, e^{-\Phi_{\text{in}}(y,\bar y)}\} = \frac{\pi}{2}\left[\epsilon(x-y) + \epsilon(\bar x - \bar y)\right] e^{-\Phi_{\text{in}}(x,\bar x)}\, e^{-\Phi_{\text{in}}(y,\bar y)}, \tag{8}
$$

where

$$
\epsilon(x) = \frac{1}{\pi}\left(x + i\sum_{n\neq 0}\frac{e^{-inx}}{n}\right), \tag{9}
$$

is the stair-step function: $\epsilon(x) = 2N+1$, for $2\pi N < x < 2\pi(N+1)$. In particular, at equal τ, this Poisson bracket vanishes. In Appendix A, we show how from (3) and (8) one derives that the *out*-field exponentials satisfy the the same causal Poisson brackets. This proves that the map (3) is canonical.

Taking into account (A.25) and its anti-chiral counterpart, we represent the map (3) as

$$
e^{-\Phi_{\text{out}}(x,\bar x)} = \mu^2\, e^{-\Phi_{\text{in}}(x,\bar x)}\,\frac{1}{4\sinh^2(\pi p)} \int_0^{2\pi} dy \int_0^{2\pi} d\bar y\, e^{2\Phi_{\text{in}}(y+x,\bar y+\bar x) - 2\pi p}. \tag{10}
$$

We will use this relation in the quantum theory to fix the operator ordering ambiguities.

With help of Equation (3), we can express the Fourier modes of the *out*-field in terms of those of the *in*-field

$$
\tilde p = -p, \quad \tilde q = -q - \log\mu^2 - Q_0(p,a,a^*) - \bar Q_0(p,\bar a,\bar a^*)
$$
$$
b_n = a_n + inQ_n(p,a,a^*), \quad \bar b_n = \bar a_n + in\bar Q_n(p,\bar a,\bar a^*), \tag{11}
$$

where

$$
Q_n(p,a,a^*) = \int_0^{2\pi}\frac{dx}{2\pi}\, e^{inx}\log A_p(x),
$$
$$
\bar Q_n(p,\bar a,\bar a^*) = \int_0^{2\pi}\frac{d\bar x}{2\pi}\, e^{in\bar x}\log \bar A_p(\bar x), \tag{12}
$$

$$
A_p(x) = \int_{-\infty}^{0} dy\, e^{py + 2a(y+x) + 2a^*(y+x)},
$$
$$
\bar A_p(\bar x) = \int_{-\infty}^{0} d\bar y\, e^{p\bar y + 2\bar a(\bar y+\bar x) + 2\bar a^*(\bar y+\bar x)}, \tag{13}
$$

$a(x)$, $\bar a(\bar x)$ are the positive frequency parts of the chiral and anti-chiral fields

$$
a(x) = i\sum_{k>0}\frac{a_k}{k}\, e^{-ikx}, \quad \bar a(\bar x) = i\sum_{k>0}\frac{\bar a_k}{k}\, e^{-ik\bar x}, \tag{14}
$$

while the complex conjugates $a^*(x)$, $\bar a^*(\bar x)$ are their negative frequency parts.

Thus, the b-modes depend on a-modes but not on the $\bar a$-modes. Similarly, $\bar b$-modes depend on $\bar a$'s but not on a's. This is an indication for the chiral structure of the S-matrix.

Finally note that

$$
Q_n(p,a,a^*) \to 0, \quad \text{when} \quad p \to 0 \tag{15}
$$

and similarly for $\bar{Q}_n$. It means that scatterings disappear at $p = 0$. This fact follows from (12) and (13), where it is convenient to use the representation of $A_p(x)$ similar to (A.25). We will use this fact in Section 4 to find recursive relations for the S-matrix.

2.2. Quantum Asymptotic Fields

We apply canonical quantization and use the standard "hat" notation for the operators corresponding to the classical counterparts.

Based on the classical picture, one has the canonical commutation relations

$$[\hat{q}, \hat{p}] = i\hbar, \quad [\hat{a}_m, \hat{a}_n] = \frac{\hbar}{2}\, m\delta_{m,-n}, \quad [\hat{\bar{a}}_m, \hat{\bar{a}}_n] = \frac{\hbar}{2}\, m\delta_{m,-n};$$
$$[\hat{\tilde{q}}, \hat{\tilde{p}}] = i\hbar, \quad [\hat{b}_m, \hat{b}_n] = \frac{\hbar}{2}\, m\delta_{m,-n}, \quad [\hat{\bar{b}}_m, \hat{\bar{b}}_n] = \frac{\hbar}{2}\, m\delta_{m,-n}, \tag{16}$$

together with

$$[\hat{a}_m, \hat{\bar{b}}_n] = [\hat{\bar{a}}_m, \hat{b}_n] = 0. \tag{17}$$

However, the commutation relations of the a-modes with the b-modes (and $\bar{a}$'s with $\bar{b}$'s) are highly non-trivial and they are defined by the quantum version of (10), which has the following form

$$: e^{-\hat{\Phi}_{\rm out}(x,\bar{x})} : = \frac{\mu_q^2}{2\sinh(\pi\hat{p})} : e^{-\hat{\Phi}_{\rm in}(x,\bar{x})} : \int_0^{2\pi} dy \times \int_0^{2\pi} d\bar{y} : e^{2\hat{\Phi}_{\rm in}(y+x,\bar{y}+\bar{x})-2\pi\hat{p}} : \frac{1}{2\sinh(\pi\hat{p})}, \tag{18}$$

with

$$\mu_q^2 = \mu^2\,\frac{\sin(\pi\hbar)}{\pi\hbar}. \tag{19}$$

The normal ordered free-field exponentials in (18) are the vertex operators defined in a standard way by separating the zero mode, the positive and the negative frequency parts, as e.g., in

$$: e^{-\hat{\Phi}_{\rm out}(x,\bar{x})} := e^{-\hat{\tilde{q}}+\hat{\tilde{p}}\tau}\, e^{-\hat{\bar{b}}^\dagger(\bar{x})}\, e^{-\hat{b}^\dagger(x)}\, e^{-\hat{\bar{b}}(\bar{x})}\, e^{-\hat{b}(x)}, \tag{20}$$

where the operators $\hat{b}(x)$, $\hat{\bar{b}}(\bar{x})$ are given similarly to (14)

$$\hat{b}(x) = i\sum_{k>0}\frac{\hat{b}_k}{k}\, e^{-ikx}, \quad \hat{\bar{b}}(\bar{x}) = i\sum_{k>0}\frac{\hat{\bar{b}}_k}{k}\, e^{-ikx}, \tag{21}$$

and $\hat{b}^\dagger(x)$, $\hat{\bar{b}}^\dagger(\bar{x})$ are their Hermitian conjugates.

The transition from the classical expression (10) to the quantum expression (18) is not unique due to operator ordering ambiguities. The requirements which lead from (10) to (18) are:

1. The spectrum of $\hat{p}$ is positive and $\hat{\tilde{p}} = -\hat{p}$, as in the classical theory.
2. The relation (4) holds in quantum theory without deformation. This assumption is in accordance with the previous one, since the asymptotic behavior of the *in*- and the *out*-field exponentials are defined by the factors $e^{-p\tau}$ and $e^{p\tau}$, respectively.
3. The conformal weight of $e^{2\,\hat{\phi}_{\rm in}(z,\bar{z})}$ is (1,1). Then, the integration in (18) gives a conformal scalar and, therefore, the *in*- and *out*-field exponentials have the same conformal weights which, due to (4), coincides with the weight of $e^{-\hat{\Phi}}$.
4. The exponentials of the asymptotic fields are Hermitian. This follows from
$$: e^{-\hat{\Phi}_{\rm in}(x,\bar{x})} :: e^{2\hat{\Phi}_{\rm in}(x+y,\bar{x}+\bar{y})-2\pi p} := : e^{2\hat{\Phi}_{\rm in}(x+y,\bar{x}+\bar{y})-2\pi p} :: e^{-\hat{\Phi}_{\rm in}(x,\bar{x})} : \tag{22}$$
which holds for $y \in (0, 2\pi)$ and $\bar{y} \in (0, 2\pi)$.
5. Conformal symmetry and Hermiticity do not fix the $\hat{p}$-dependent factors, since $\hat{p}$ is a conformal scalar. They are determined via the locality condition for the Liouville field
$$[e^{-\hat{\Phi}(\tau,\sigma)}, e^{-\hat{\Phi}(\tau,\sigma')}] = 0. \tag{23}$$
With the chosen ordering they are the same as in the classical expression (10).
6. Finally, the requirement that the quantum Liouville field satisfies Equation (1) determines the overall factor as in (19).

For the technical details related to these requirements and how they lead from (10) to (18), we refer to refs. [1, 9–11].

3. Equation for the S-Matrix

In this section, we introduce the coherent states of the asymptotic fields as the generators for the *in* and *out* Fock spaces and use the operator relation (18) to derive the equation for the transition amplitude from the coherent *in* state to the coherent *out* state. This defines the S-matrix of the theory. We then show how to obtain the reflection amplitude and Fock space transition amplitudes.

Consider the momentum dependent asymptotic vacuum states: $|p\rangle_{\rm in}$, with $p > 0$, and $|\tilde{p}\rangle_{\rm out}$, with $\tilde{p} < 0$. Motivated by the classical situation we *assume* that the states $|p\rangle_{\rm in}$ are annihilated not only by $(\hat{a}(x), \hat{\bar{a}}(\bar{x}))$, but also by $(\hat{b}(x), \hat{\bar{b}}(\bar{x}))$, and likewise for $|\tilde{p}\rangle_{\rm out}$. Therefore, these states are related by

$$|p\rangle_{\rm in} = R(p)\,|-p\rangle_{\rm out}, \tag{24}$$

where $R(p)$ is called the reflection amplitude, and one has

$${}_{\rm out}\langle\tilde{p}\,|\,p\rangle_{\rm in} = \delta(\tilde{p}+p)\,R(p). \tag{25}$$

Since apart from the zero modes the two chiral sectors decouple, it suffices to discuss one of them; the other then follows immediately.

We introduce the asymptotic coherent states

$$|p, a\rangle_{\rm in} = \exp\left(\frac{2}{\hbar}\sum_{k>0}\frac{a_k}{k}\,\hat{a}_k^\dagger\right)|\,p\rangle_{\rm in},$$
$${}_{\rm out}\langle\tilde{p},\, b^*| = {}_{\rm out}\langle\tilde{p}\,|\exp\left(\frac{2}{\hbar}\sum_{k>0}\frac{b_k^*}{k}\,\hat{b}_k\right), \tag{26}$$

and the corresponding transition amplitude ${}_{\rm out}\langle \tilde{p}, b^*|p,a\rangle_{\rm in}$. Due to $\hat{\tilde{p}} + \hat{p} = 0$ it has the structure

$$ {}_{\rm out}\langle \tilde{p}, b^*|p,a\rangle_{\rm in} = \delta(\tilde{p}+p)\, R(p)\, S_p[b^*,a], \tag{27}$$

where $S_p[b^*,a]$ is a holomorphic function of the modes (b_k^*, a_k), for $k>0$ and contains p as a parameter. Note that from (24) follows that $S_p[0,a] = S_p[b^*,0] = 1$.

One can associate to the function $S_p[b^*,a]$ an operator $\hat{S}_p$ via

$$ {}_{\rm in}\langle p', b^*|\hat{S}_p|p,a\rangle_{\rm in} = \delta(p'-p)\, S_p[b^*,a]. \tag{28}$$

where $\hat{S}_p$ is a function of the *in*-field mode operators $\hat{a}_n$.[12]

We will now derive equations for $R(p)$ and $S_p[b^*,a]$, using the operator relation (18). As to $S_p[b^*,a]$ it is convenient to indicate the modes numbers and write it as $S_p(b_k^*, a_k)$.

From the canonical commutation relations (16) follows:

$$ {}_{\rm out}\langle \tilde{p}, b^*|e^{-\hat{\tilde{q}}} = {}_{\rm out}\langle \tilde{p}-i\hbar, b^*|, \quad e^{\hat{q}}|p,a\rangle_{\rm in} = |p-i\hbar, a\rangle_{\rm in}, \tag{29}$$

and also the standard properties of the coherent states (26)

$$\begin{aligned} &\hat{a}_k\,|p,a\rangle = a_k\,|p,a\rangle, && \hat{a}_k^\dagger\,|p,a\rangle = \frac{\hbar k}{2}\frac{\partial}{\partial a_k}\,|p,a\rangle, \\ &\langle b^*,\tilde{p}|\,\hat{b}_k^\dagger = b_m^*\,\langle b^*,\tilde{p}|, && \langle b^*,\tilde{p}|\,\hat{b}_k = \frac{\hbar k}{2}\frac{\partial}{\partial b_k^*}\,\langle b^*,\tilde{p}|. \end{aligned} \tag{30}$$

Projecting the operator relation (18) between the coherent states (26), we find for the left hand side

$$\begin{aligned} &{}_{\rm out}\langle \tilde{p}, b^*| : e^{-\hat{\Phi}_{\rm out}(x,\bar{x})} : |p,a\rangle_{\rm in} \\ &\quad = \delta(\tilde{p}+p-i\hbar)\, e^{(p-i\hbar/2)\tau}\, R(p)\, e^{-b^*(x)}\, S_p\Big(b_k^* - \tfrac{i\hbar}{2}\,e^{-ikx}, a_k\Big). \end{aligned} \tag{31}$$

To calculate the right hand side one has to use (29) and (30) together with the exchange relation

$$ e^{-\hat{a}(x)}\, e^{2\hat{a}^\dagger(x+\gamma)} = e^{2\hat{a}^\dagger(x+\gamma)}\, e^{-\hat{a}(x)}\big(1-e^{i\gamma}\big)^\hbar. \tag{32}$$

This leads to

$$\begin{aligned} &\delta(\tilde{p}+p-i\hbar)\, e^{(p-i\hbar/2)\tau}\, \frac{\mu_q^2\, R(p-i\hbar)}{4\sinh(\pi p)\,\sinh(\pi p - i\pi\hbar)} \\ &\quad \times e^{-a(x)} \int_0^{2\pi} d\tilde{\gamma}\, e^{(p-i\hbar)(\tilde{\gamma}-\pi)}\big(1-e^{i\tilde{\gamma}}\big)^\hbar \int_0^{2\pi} d\gamma\, e^{(p-i\hbar)(\gamma-\pi)}\big(1-e^{i\gamma}\big)^\hbar \\ &\quad \times e^{2a(x+\gamma)}\, S_{p-i\hbar}\Big(b_k^*, a_k + \tfrac{i\hbar}{2}\,e^{ikx} - i\hbar\, e^{ik(x+\gamma)}\Big). \end{aligned} \tag{33}$$

For vanishing non-zero modes, $a_k = b_k^* = 0$, we obtain from (31) and (33) the following equation for $R(p)$:

$$ R(p) = \frac{\pi^2\mu_q^2\,\gamma_p^2}{\sinh(\pi p)\,\sinh(\pi(p-i\hbar))}\, R(p-i\hbar), \tag{34}$$

where γ_p is the integral

$$ \gamma_p := \int_0^{2\pi}\frac{d\gamma}{2\pi}\, e^{(p-i\hbar)(\gamma-\pi)}\big(1-e^{i\gamma}\big)^\hbar = \frac{\Gamma(1+\hbar)}{\Gamma(1-ip)\Gamma(1+\hbar+ip)}. \tag{35}$$

Using in addition the identity $\sinh(\pi p)\,\Gamma(1+ip)\,\Gamma(1-ip) = \pi p$, (34) can be written as

$$ R(p) = \mu_q^2\,\frac{\Gamma(1+ip)\Gamma(1-\hbar-ip)\Gamma^2(1+\hbar)}{\Gamma(1-ip)\Gamma(1+\hbar+ip)p(p-i\hbar)}\, R(p-i\hbar). \tag{36}$$

Its solution is

$$ R(p) = -\Big(\mu_q^2\,\Gamma^2(\hbar)\Big)^{-\frac{ip}{\hbar}}\,\frac{\Gamma(ip/\hbar)\Gamma(ip)}{\Gamma(-ip/\hbar)\Gamma(-ip)}. \tag{37}$$

This is the unique solution if we require that it has the correct semiclassical limit $R = \exp(i/\hbar F^{(0)})$ with $F^{(0)} = 2p[\log(p/\mu) - 1]$. The reflection amplitude corresponds to the 2-point function of Liouville theory.[12,13]

The equation for $S_p(b_k^*, a_k)$ then reduces to

$$\begin{aligned} &e^{-b^*(x)}\, S_p\Big(b_k^* - \frac{i\hbar}{2}\,e^{-ikx}, a_k\Big) \\ &\quad = \frac{e^{-a(x)}}{\gamma_p}\int_0^{2\pi}\frac{d\gamma}{2\pi}\, e^{(p-i\hbar)(\gamma-\pi)}\big(1-e^{i\gamma}\big)^\hbar e^{2a(x+\gamma)} \\ &\qquad \times S_{p-i\hbar}\Big(b_k^*, a_k + \frac{i\hbar}{2}\,e^{ikx} - i\hbar\, e^{ik(x+\gamma)}\Big). \end{aligned} \tag{38}$$

We expand $S_p(b_k^*, a_k)$ in powers of the modes as

$$\begin{aligned} S_p(b^*,a) &= 1 + S_{-1,1}(p)\, b_1^* a_1 + S_{-2,2}(p)\, b_2^* a_2 + S_{-1,-1,2}(p)\, b_1^{*\,2} a_2 \\ &\quad + S_{-2,1,1}(p)\, b_2^* a_1^2 + S_{-1,-1,1,1}(p)\, b_1^{*\,2} a_1^2 + \cdots, \end{aligned} \tag{39}$$

where the p-dependent coefficients define the transition amplitudes between the Fock space states, which are specified by their indices and one can read them off from (26). As a result of energy conservation the transition coefficients are non-zero only when the sum of the indices is zero.[6]

The first and the second level transition amplitudes were obtained in ref. [12], using the conformal symmetry of the S-matrix. The same results were reproduced by a slightly different method in ref. [6], where some higher level transitions were also computed. Here, we demonstrate how the transition amplitudes can be obtained from (38). Setting in (38) $b_k^* = 0$, for all $k>0$, and comparing the terms linear in a_1, we find the relation

$$ \frac{i\hbar}{2}\, S_{-1,1}(p) = i + 2i\frac{\gamma_{p-i}}{\gamma_p}, \tag{40}$$

where $S_{-1,1}(p)$ is the first level transition amplitude in (39). Using (35) this yields

$$ S_{-1,1}(p) = \frac{2}{\hbar}\,\frac{1+\hbar-ip}{1+\hbar+ip}, \tag{41}$$

which coincides with the result obtained in refs. [6, 12]. The same expression is obtained by setting $a_k = 0$ in (38) and comparing the terms linear in b_1^*.

One can repeat the same procedure at higher levels. At the second level, one can still obtain all amplitudes by studying Equation (38) at $a = 0$ or at $b^* = 0$, if one combines it with time-reversal symmetry which relates e.g., $S_{-2,1,1}$ to $S_{-1,-1,2}$. Doing this one finds the known results. At higher levels one also needs to consider (38) for a and b^* both nonzero. In this case one finds more complicated equations which relate matrix elements with shifted momenta. We will not study these equations here as our main goal is to solve Equation (38).

4. Recursive Relations for the Scattering Matrix

The scattering operator $\hat{S}_p$ defined in (28) describes Fock space transition amplitudes. It contains the momentum p as a parameter. For discrete momenta $p = i\hbar N$, where N is a positive integer, we will construct the normal symbol of this operator as a multiple contour integral. In this way, we recover the representation, which we had previously derived in ref. [7] from a conjectured functional integral representation of the generating functional of S-matrix elements, thus proving its validity, at least for these infinitely many discrete momentum values.

To this end we introduce

$$U_p[b^*, a] = \exp\left(-\frac{2}{\hbar}\sum_{k>0}\frac{b_k^*\, a_k}{k}\right) S_p[b^*, a]. \tag{42}$$

The function $U_p(b_k^*, a_k)$ is also holomorphic in the modes (b_k^*, a_k). At $b_k^* = a_k^*$, it coincides with the normal symbol[2] of the S-matrix.

From (38), we obtain the following equation for $U_p(b_k^*, a_k)$

$$U_p(b_k^*, a_k) = \frac{1}{\gamma_p}\int_0^{2\pi}\frac{dy}{2\pi}\, e^{(p-i\hbar)(y-\pi)}\, e^{2[a(x+y)+b^*(x+y)]} \times U_{p-i\hbar}\left(b_k^* + \tfrac{i\hbar}{2}\, e^{-ikx}, a_k + \tfrac{i\hbar}{2}\, e^{ikx} - i\hbar\, e^{ik(x+y)}\right). \tag{43}$$

where we have shifted the b_k^* modes by $b_k^* \mapsto b_k^* + \frac{i\hbar}{2}\, e^{-ikx}$.

At $p = i\hbar N$, $N \in \mathbb{N}_+$, it follows from (43) (with $a_n = b_{-n}^*$ for $n < 0$)

$$U_{i\hbar N}(a_n) = \frac{1}{\gamma_{i\hbar N}}\int_0^{2\pi}\frac{dy}{2\pi}\, e^{i(N-1)\hbar(y-\pi)}\exp\left(2i\sum_{n\neq 0}\frac{a_n}{n}e^{-in(x+y)}\right) \times U_{i\hbar(N-1)}\left(a_n + \tfrac{i\hbar}{2}\, e^{inx} - i\hbar\,\theta_n\, e^{in(x+y)}\right), \tag{44}$$

where $\theta_n = 0$, for $n < 0$ and $\theta_n = 1$, for $n > 0$. This can be treated as a recursive relation which we will solve.

For this, we assume that $U_0(a_n) = 1$, i.e., at $p = 0$ there is no scattering, i.e., $\hat{S}_{p=0} = \mathbb{1}$. This assumption is physically reasonable; it is supported by the classical picture (see the remark at the end of Section 2.1) and by the known exact quantum results at low levels. From (44) and (35), we then find

$$U_{i\hbar}(a_n) = \int_0^{2\pi}\frac{dy}{2\pi}\exp\left(2i\sum_{n\neq 0}\frac{a_n}{n}e^{-in(x+y)}\right) = 1 + \sum_{\nu\geq 2}\frac{(2i)^\nu}{\nu!}\sum_{n_1,\ldots n_\nu}\frac{a_{n_1}\ldots a_{n_\nu}}{n_1\cdots n_\nu}\,\delta_{n_1+\cdots+n_\nu}. \tag{45}$$

As expected, the right-hand side is x-independent and agrees with the result, which was obtained in ref. [7] from the path integral representation of the S-matrix.

To proceed with the analysis of the recursive relations, it is convenient to introduce complex variables on the unit circle: $\xi = e^{iy}$, $\zeta = e^{ix}$ and rewrite (44) as

$$U_{i\hbar N}(a_n) = \frac{e^{-i\pi\hbar(N-1)}}{\gamma_{i\hbar N}}\oint\frac{d\xi}{2\pi i\xi}\,\xi^{\hbar(N-1)}\exp\left(2i\sum_{n\neq 0}\frac{a_n}{n}(\xi\zeta)^{-n}\right) \times U_{i\hbar(N-1)}\left(a_n + \tfrac{i\hbar}{2}\,\zeta^n - i\hbar\,\theta_n(\xi\zeta)^n\right), \tag{46}$$

Then, at $N = 2$, from (45) and (46), we find

$$U_{2i\hbar}(a_n) = \frac{e^{-i\pi\hbar}}{\gamma_{2i\hbar}}\oint\frac{d\xi}{2\pi i\xi}\xi^\hbar\exp\left(2i\sum_{n\neq 0}\frac{a_n}{n}(\xi\zeta)^{-n}\right) \times\oint\frac{d\zeta_1}{2\pi i\zeta_1}\exp\left(2i\sum_{n\neq 0}\frac{a_n}{n}\zeta_1^{-n}\right)\left(1-\frac{\zeta}{\zeta_1}\right)^\hbar \times\left(1-\frac{\zeta_1}{\zeta}\right)^{-\hbar}\left(1-\frac{\zeta\xi}{\zeta_1}\right)^{-2\hbar}, \tag{47}$$

with $\zeta_1 = \xi_1\zeta$. Introducing a new contour variable $\zeta_2 = \xi\zeta$, from (47) one obtains

$$U_{2i\hbar}(a_n) = \frac{e^{-i\pi\hbar}}{\gamma_{2i\hbar}}\oint\frac{d\zeta_1}{2\pi i\zeta_1}\oint\frac{d\zeta_2}{2\pi i\zeta_2}\exp\left(2i\sum_{n\neq 0}\frac{a_n}{n}(\zeta_1^{-n}+\zeta_2^{-n})\right) \times(\zeta_1\zeta_2)^\hbar(\zeta_1-\zeta_2)^{-2\hbar}. \tag{48}$$

The recursive relation (46) and the calculation similar to (47) for general N leads to

$$U_{i\hbar N}(a_n) = \prod_{\alpha=1}^{N}\frac{1}{\gamma_{i\hbar\alpha}}\oint\frac{d\zeta_1}{2\pi i\zeta_1}\cdots\oint\frac{d\zeta_N}{2\pi i\zeta_N} \times\exp\left(2i\sum_{n\neq 0}\frac{a_n}{n}(\zeta_1^{-n}+\cdots\zeta_N^{-n})\right) \times\prod_{1\leq\alpha<\beta\leq N}(\zeta_\alpha\zeta_\beta)^\hbar(\zeta_\alpha-\zeta_\beta)^{-2\hbar}. \tag{49}$$

This agrees with the expression that we derived in ref. [7] from a functional integral representation of $S_p(b^*, a)$, which is based

[2] Given the operator $:\hat{U}(\hat{a}^\dagger, \hat{a},):$, whose matrix elements between coherent *in*-states reproduce those of $\hat{S}$, we have ${}_{\rm in}\langle b^*| :\hat{U}(\hat{a}^\dagger, \hat{a}): |a\rangle_{\rm in} = {}_{\rm in}\langle b^*|a\rangle_{\rm in}\, U(b^*, a) = S(b^*, a)$ with ${}_{\rm in}\langle b^*|a\rangle_{\rm in} = \exp(\frac{2}{\hbar}\sum_{k>0}\frac{b_k^* a_k}{k})$. $U(a^*, a)$ is called the normal symbol of $\hat{S}$ and the operator $:\hat{U}:$ can be reconstructed from it.

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on a one-dimensional field theory on a circle. It was proposed for arbitrary values of the momenta and was shown to reduce to (49) for $p = i\hbar N$. Expanding (49) in powers of a_n one obtains integral representations of the S-matrix elements. For low values of N one can perform the integrals as we have demonstrated in ref. [7]. Here, we will not attempt to go beyond what was obtained there, as the main purpose of this note is to prove the functional integral representation for $S_p(b^*, a)$ introduced in ref. [6, 7]. This was achieved for an infinite set of discrete momenta.

Given the above expression for the normal symbol of the S-matrix, we can explicitly check the assumption $U_0(a) = 1$. This can indeed be done using the generalization of the Dotsenko–Fateev integrals[14] found in ref. [15] and the explicit form of γ_p given in (35). Then, within the functional integral representation of refs. [6, 7], the first factor in (49) becomes the vacuum-to-vacuum amplitude for the one-dimensional field theory, while the second factor, the multiple integral, describes the non-trivial scattering.[3]

Once the S-matrix is known for an infinite set of discrete momenta, one can attempt to analytically continued to physical, i.e., real positive momenta. For this a further generalization of the Dotsenko–Fateev integrals is required, which is at present not known. What can be obtained, however, is the analytic continuation of the vacuum-to-vacuum amplitude to real momenta which leads to a representation in terms of a single integral. This is shown in Appendix B.

Appendix A: Poisson Brackets of the Chiral Fields

Here, we derive the Poisson brackets algebra of the chiral fields using a new form of the screening charge given by (A.3). The algebra is then used to check the canonicity of the map (3) and to prepare the system for quantization.

Let us consider the chiral free field

$$\phi(x) = q + \tfrac{1}{2}\, p\, x + i \sum_{n\neq 0} \frac{a_n}{n}\, e^{-inx}. \tag{A.1}$$

Its Fourier modes satisfy the canonical relations (6) and one gets the Poisson brackets

$$\{e^{-\phi(x)}, e^{-\phi(y)}\} = \frac{\pi}{2}\, e^{-\phi(x)}\, e^{-\phi(y)}\, \epsilon(x-y). \tag{A.2}$$

As for the Liouville *in*-field, we assume that $p > 0$ and introduce the screening charge

$$A(x) = \int_{-\infty}^{x} dz\; e^{2\phi(z)}. \tag{A.3}$$

This chiral field is quasi-periodic like the exponential $e^{2\phi(x)}$:

$$e^{2\phi(x+2\pi)} = e^{2\pi p}\, e^{2\phi(x)}, \quad A(x+2\pi) = e^{2\pi p}\, A(x). \tag{A.4}$$

[3] We stress that the vacuum-to-vacuum amplitude of Liouville theory is given by the reflection amplitude $R(p)$. The vacuum-to-vacuum amplitude we have in mind here is the one corresponding to the functional integral representation of $S_p[b^*, a]$ of refs. [6, 7].

We also introduce the chiral field

$$\chi(x) = e^{-\phi(x)}\, A(x), \tag{A.5}$$

and our aim is to derive the Poisson brackets algebra of these chiral fields.

First, we calculate the Poisson brackets $\{e^{-\phi(x)}, A(y)\}$, which by Equations (A.2) and (A.3) reads

$$\{e^{-\phi(x)}, A(y)\} = \pi\, e^{-\phi(x)} \int_{-\infty}^{y} dz\; e^{2\phi(z)}\, \epsilon(z-x). \tag{A.6}$$

Using the relations $A'(z) = e^{2\phi(z)}$ and

$$\epsilon'(z-x) = 2 \sum_{k=-\infty}^{\infty} \delta(z - x - 2\pi k), \tag{A.7}$$

we rewrite the integral term in (A.6) as follows

$$\int_{-\infty}^{y} dz\; A'(z)\epsilon(z-x) = A(y)\epsilon(y-x) - 2\int_{-\infty}^{y} dz\; A(z) \times \sum_{k=-\infty}^{\infty} \delta(z-x-2\pi k). \tag{A.8}$$

The integration of δ-functions here, together with (A.4), yields

$$\int_{-\infty}^{y} dz\; A(z) \sum_{k=-\infty}^{\infty} \delta(z-x-2\pi k) = A(x) \sum_{k=-\infty}^{N} e^{2\pi pk}, \tag{A.9}$$

where N is defined by the inequalities $y - 2\pi < x + 2\pi N < y$, i.e.,

$$2\pi N < y - x < 2\pi(N+1). \tag{A.10}$$

Then, $2N + 1 = \epsilon(y-x)$ and the sum of the geometric progression in (A.9) is represented as

$$\sum_{k=-\infty}^{N} e^{2\pi pk} = \frac{e^{2\pi pN}}{1 - e^{-2\pi p}} = \frac{e^{\pi p\,\epsilon(y-x)}}{2\sinh \pi p} \equiv \theta_p(y-x). \tag{A.11}$$

As a result, the Poisson bracket (A.6) becomes

$$\{e^{-\phi(x)}, A(y)\} = \pi\, e^{-\phi(x)} \big[A(y)\,\epsilon(y-x) - 2A(x)\,\theta_p(y-x)\big]. \tag{A.12}$$

Note that (A.12) is regular at $y = x$ and one has

$$\{e^{-\phi(x)}, A(x)\} = -\pi\, \coth(\pi p)\, e^{-\phi(x)}\, A(x). \tag{A.13}$$

Now, we calculate the Poisson brackets $\{A(x), A(y)\}$ given by the double integral

$$\{A(x), A(y)\} = 2\pi \int_{-\infty}^{x} du\; e^{2\phi(u)} \int_{-\infty}^{y} dz\; e^{2\phi(z)}\, \epsilon(u-z). \tag{A.14}$$

Here, the z-integration is similar to the one given in (A.6) and we end up with the u-integral

$$\{A(x), A(y)\} = 2\pi \int_{-\infty}^{x} du\; e^{2\phi(u)} \big[A(y)\,\epsilon(u-y) + 2A(u)\,\theta_p(y-u)\big]. \tag{A.15}$$

The integration of the first term is again similar to (A.6) and, together with a partial integration of the second term, we obtain

$$\{A(x), A(y)\} = 2\pi[A(x)A(y)\,\epsilon(x-y) - 2A^2(y)\,\theta_p(x-y) + A^2(x)\,\theta_p(y-x)] + 2\pi\int_{-\infty}^{x} du\; A^2(u)\,\theta_p'(y-u). \tag{A.16}$$

Since $\theta_p(z)$ is a stair-step function, one gets

$$\theta_p'(y-u) = \sum_{k=-\infty}^{\infty} e^{-2\pi pk}\,\delta(y-u+2\pi k), \tag{A.17}$$

and the integration of the δ-functions in the last term of (A.16) yields

$$\int_{-\infty}^{x} du\; A^2(u)\sum_{k=-\infty}^{\infty} e^{-2\pi pk}\,\delta(y-u+2\pi k) = A^2(y)\sum_{k=-\infty}^{M} e^{2\pi pk} = A^2(y)\,\theta_p(x-y), \tag{A.18}$$

similarly to (A.11). This leads to

$$\{A(x), A(y)\} = 2\pi\left[A(x)A(y)\,\epsilon(x-y) - A^2(y)\,\theta_p(x-y) + A^2(x)\,\theta_p(y-x)\right]. \tag{A.19}$$

Finally, using (A.6) and (A.19), it is straightforward to check that

$$\{\chi(x), \chi(y)\} = \frac{\pi}{2}\,\chi(x)\,\chi(y)\,\epsilon(x-y). \tag{A.20}$$

We now introduce the anti-chiral free field

$$\bar\phi(\bar x) = \bar q + \tfrac{1}{2}\,\bar p\bar x + i\sum_{n\neq 0}\frac{\bar a_n}{n}\,e^{-in\bar x}, \tag{A.21}$$

where $\{\bar q, \bar p\} = 1$ and the non-zero modes satisfy the canonical relations (6). The anti-chiral fields $\bar A(\bar x)$ and $\bar\chi(\bar x)$ are defined as the chiral ones and we similarly get

$$\{\bar\chi(\bar x), \bar\chi(\bar y)\} = \frac{\pi}{2}\,\bar\chi(x)\,\bar\chi(\bar y)\,\epsilon(\bar x-\bar y). \tag{A.22}$$

The Poisson brackets, (A.20) and (A.22), are used to compute the Poisson brackets for the *out*-field exponentials, which are given as functionals of the *in*-field, cf. (3). One remaining issue is the treatment of the zero modes. Note that the *in*-field of Liouville theory (2) is given as the sum of the chiral (A.1) and the anti-chiral (A.21) free-fields

$$\Phi_{\text{in}}(x,\bar x) = \phi(x) + \bar\phi(\bar x), \tag{A.23}$$

restricted by the constraints $\bar p - p = 0$, $\bar q = 0$. Therefore, the Poisson bracket for the *in*-field functionals can be calculated by the Dirac brackets defined by these two second class constraints. However, since the coordinate zero mode of Φ_{in} is $q+\bar q$ and it has vanishing Poisson brackets with the constraint $\bar p - p$, the Dirac brackets of the functions of Φ_{in} coincide with their Poisson brackets treated with independent chiral and anti-chiral zero-modes. Therefore, the Poisson brackets of the *out*-field exponentials (8) directly follow from (A.20) and (A.22).

Coming back to the Poisson brackets at coincident points (A.13), we obtain

$$\{e^{-\phi(x)}, \sinh(\pi p)\,A(x)\} = 0. \tag{A.24}$$

We use its quantum analogue in Section 2.1 to fix the operator ordering ambiguities for the construction of the *out*-field exponential (18).

Indeed, due to the monodromy (A.4), the chiral field (A.3) is represented as

$$A(x) = \sum_{n=0}^{\infty}\int_{-2\pi(n+1)}^{-2\pi n} dy\, e^{2\phi(y+x)} = \frac{1}{2\sinh(\pi p)}\int_0^{2\pi} dy\, e^{2\phi(y+x)-\pi p}, \tag{A.25}$$

and by (A.24) one gets

$$\left\{e^{-\phi(x)}, \int_0^{2\pi} dy\, e^{2\phi(y+x)-\pi p}\right\} = 0. \tag{A.26}$$

Note that for $y\in(0,2\pi)$ one has the vanishing Poisson brackets also with the integrand

$$\{e^{-\phi(x)}, e^{2\phi(y+x)-\pi p}\} = 0. \tag{A.27}$$

Its quantum version is used in Equation (22), which provides Hermiticity of the *out*-field vertex operator in Equation (18).

Appendix B: The Vacuum-to-Vacuum Amplitude

For a positive integer N we introduce the function

$$Z_N(\hbar) \equiv \prod_{k=1}^{N}\gamma_{i\hbar k} = \prod_{k=1}^{N}\frac{\Gamma(1+\hbar)}{\Gamma(1+k\hbar)\Gamma(1+\hbar-k\hbar)}. \tag{B.1}$$

Our aim is to find its analytical continuation in N.

We use the integral representation

$$\log\Gamma(z) = \int_0^\infty \frac{dt}{t}\left[\frac{e^{-zt}-e^{-t}}{1-e^{-t}} + (z-1)e^{-t}\right], \tag{B.2}$$

and rewrite (B.1) as $Z_N(\hbar) = e^{I_N(\hbar)}$, with

$$I_N(\hbar) = \sum_{k=1}^{N}\int_0^\infty\frac{dt}{t}\left[\frac{e^{-(1+\hbar)t}+e^{-t}-e^{-(1+k\hbar)t}-e^{-(1+\hbar-k\hbar)t}}{1-e^{-t}}\right]. \tag{B.3}$$

After the summation we get an analytical dependence on N

$$I_N(\hbar) = \int_0^\infty\frac{dt}{t}\left[N\frac{e^{-(1+\hbar)t}+e^{-t}}{1-e^{-t}} - \frac{e^{-t}}{1-e^{-t}}\left(\frac{e^{-\hbar t}(1-e^{-N\hbar t})}{1-e^{-\hbar t}} + \frac{1-e^{N\hbar t}}{1-e^{\hbar t}}\right)\right], \tag{B.4}$$

and for $N\hbar = -ip$ we find

$$I(p,\hbar) = -\frac{i}{\hbar}\int_0^\infty \frac{dt}{t(e^t-1)}\left[p\left(1+e^{-\hbar t}\right) + i\hbar\frac{e^{ipt}-e^{-ipt}}{e^{\hbar t}-1}\right]. \tag{B.5}$$

The analytical continuation described here is similar to the one used in [13] for the 3-point function of Liouville theory.

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Conflict of Interest

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